Minimal Equational Theories for Quantum Circuits

LICS'24 39th Annual ACM/IEEE Symposium on Logic in Computer Science

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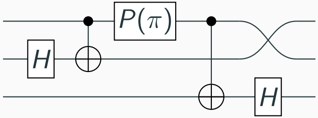
What is it all about?

Quantum circuits are a rigourous graphical language used to represent quantum algorithms.



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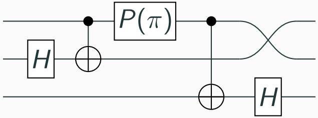

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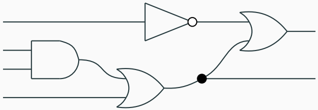

Quantum circuits as a graphical language

Quantum circuits are generated by

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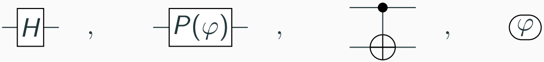

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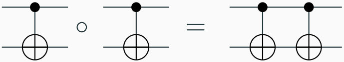

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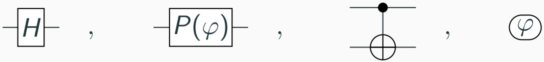

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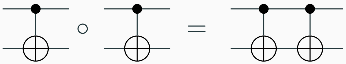

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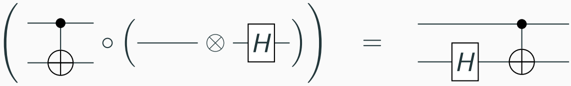


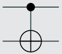
Standard interpretation of quantum circuits

Interpretation

$$[\![C_2 \circ C_1]\!] = [\![C_2]\!] \circ [\![C_1]\!] \qquad [\![C_1 \otimes C_2]\!] = [\![C_1]\!] \otimes [\![C_2]\!]$$

$$[\![\ \boxed{}\ ]\!] = (1) \qquad [\![\ \varphi\ ]\!] = (e^{i\varphi})$$

$$[\![\ \text{——}\ ]\!] = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad [\![\ -\boxed{H}-\ ]\!] = 1/\sqrt{2} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \qquad [\![\ -\boxed{P(\varphi)}-\ ]\!] = \begin{pmatrix} 1 & 0 \\ 0 & e^{i\varphi} \end{pmatrix}$$

$[\![$  $]\!] = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$ $\qquad$ $[\![$  $]\!] = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$

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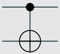
$[\![$  $]\!] = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$ $\qquad$ $[\![$  $]\!] = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$

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Quantum circuits as a graphical language

Formally, quantum circuits are defined as a symmetric monoidal category, which ensure some deformation equations such that

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This framework captures the intuitive behaviour of wires by ensuring that circuits are defined "up to deformation".

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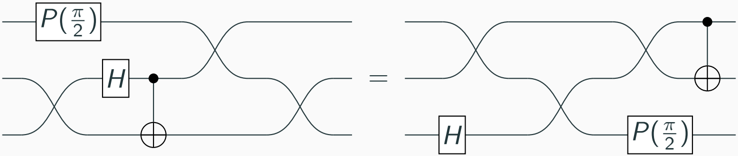
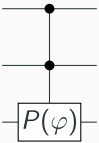


Controlled gates as shortcut notations

We use the standard bullet notation for controlled gates.



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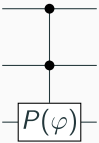

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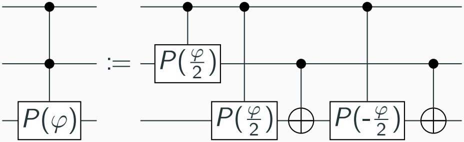
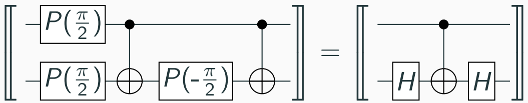


Motivations

Distinct circuits can have the same interpretation.



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Given a quantum algorithm, which circuit is the best?

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- Resource optimisation (for instance the number of gates).
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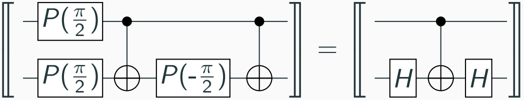

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Using equations to transform circuits

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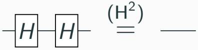

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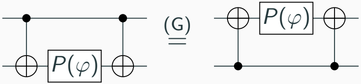

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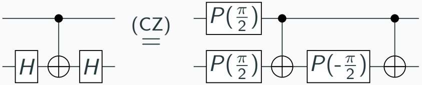

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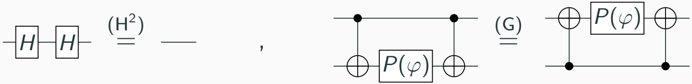

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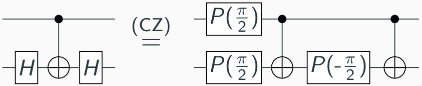

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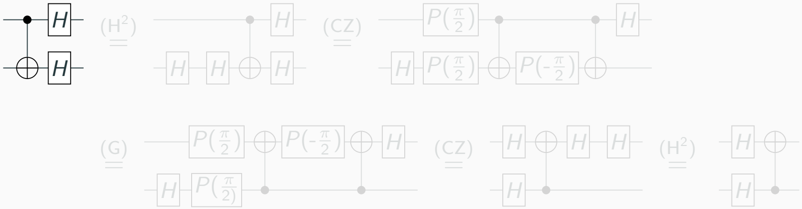

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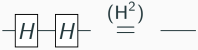

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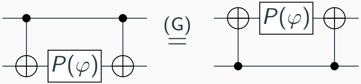

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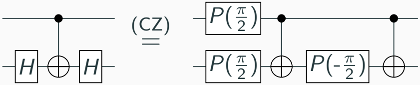

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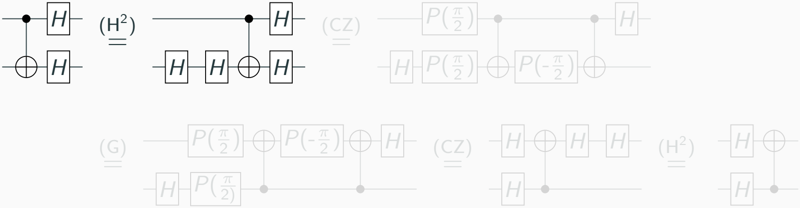

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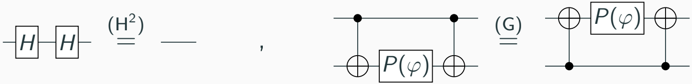

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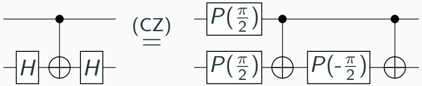

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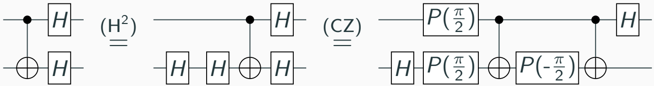

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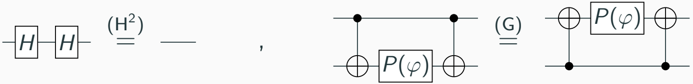

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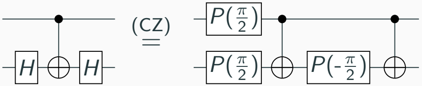

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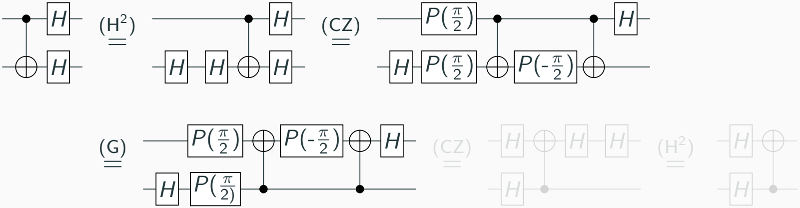

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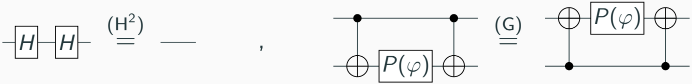

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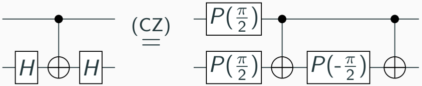

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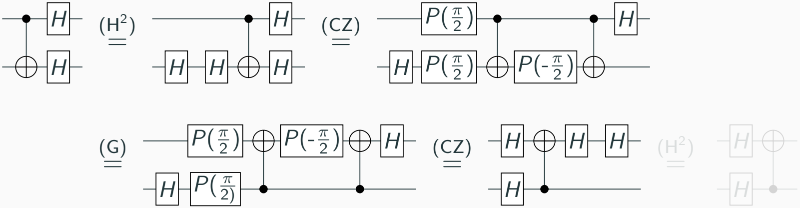

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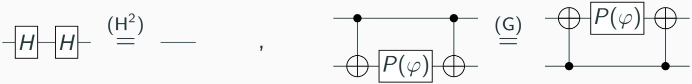

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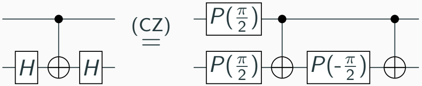

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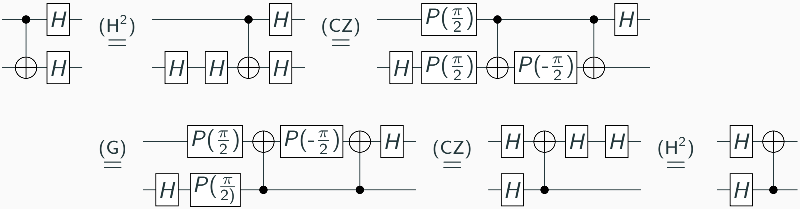

Complete and sound equational theory

Is there an equational theory (i.e. a set of axioms) Γ from which we can derive any true equation and only true equations?

Soundness

Any derivable equation is true.

$$\forall C_1, C_2 \quad : \quad \Gamma \vdash C_1 = C_2 \implies [\![C_1]\!] = [\![C_2]\!]$$

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Any true equation is derivable.

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Previous work [Clément,Heurtel,Mansfield,Perdrix,Valiron LICS'23]:

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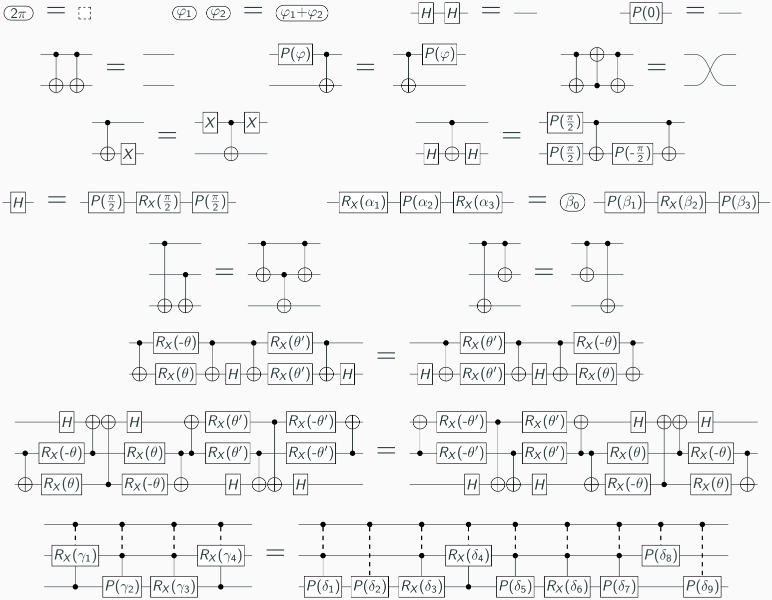
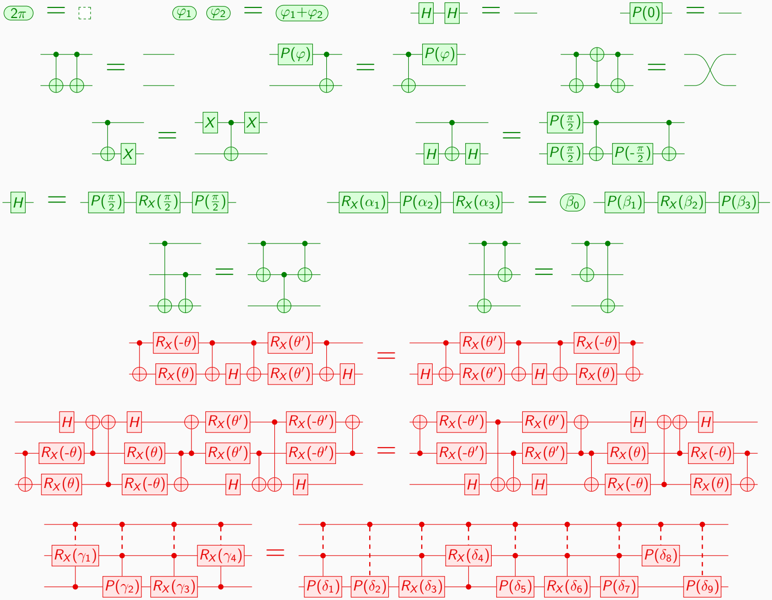


Some easy and some intricate equations



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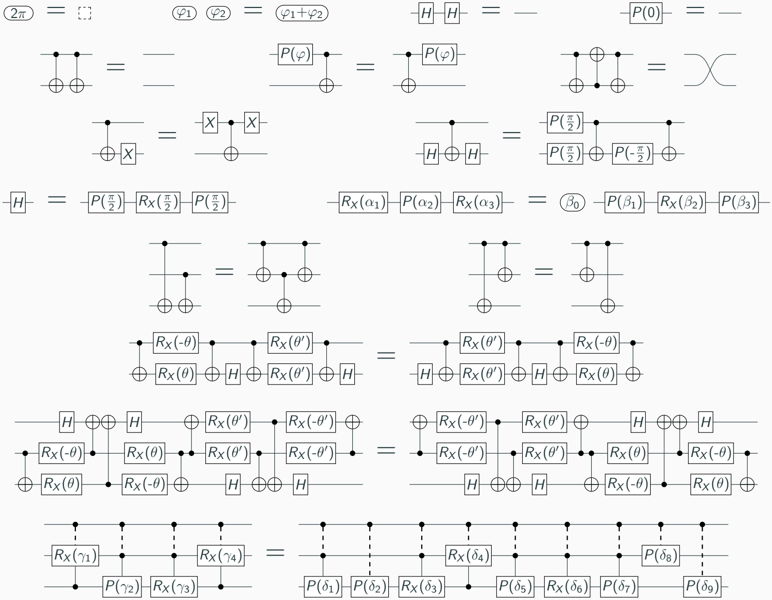

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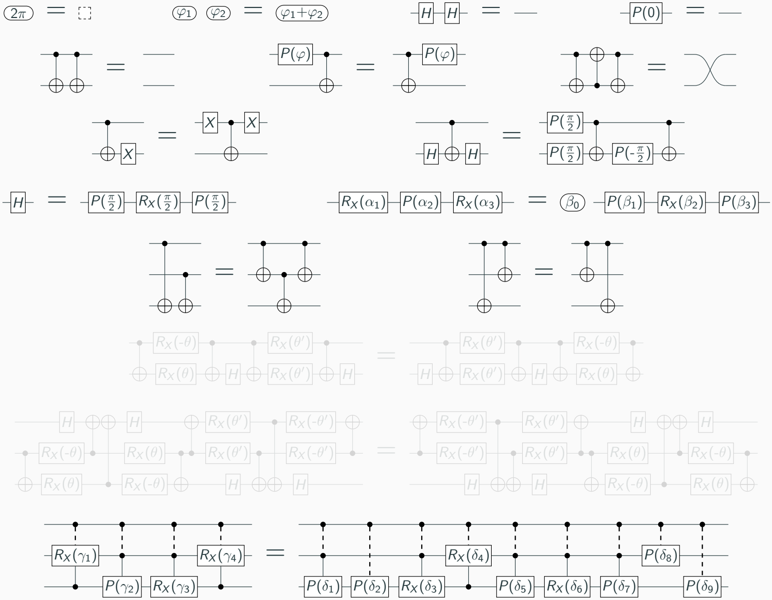

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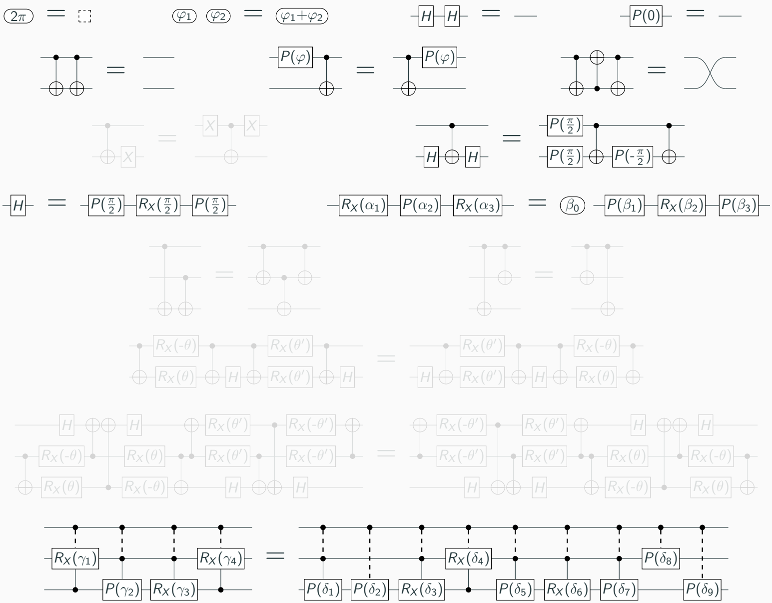

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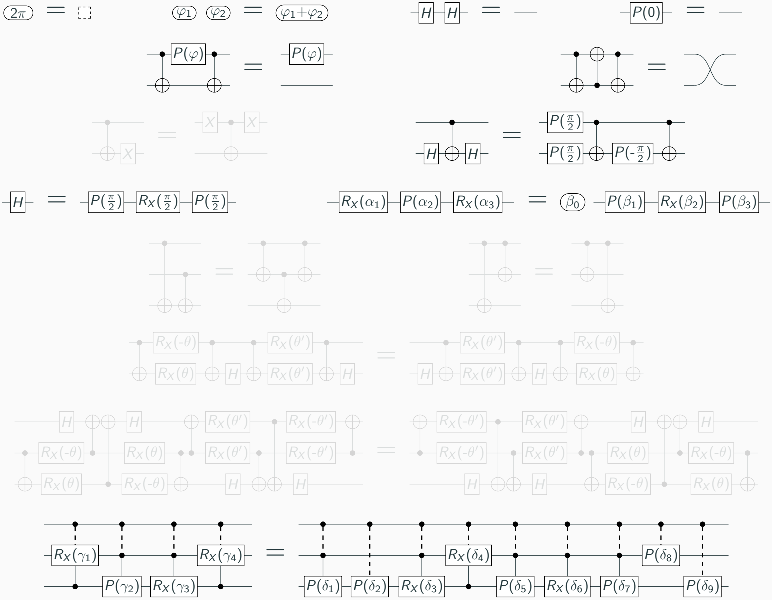

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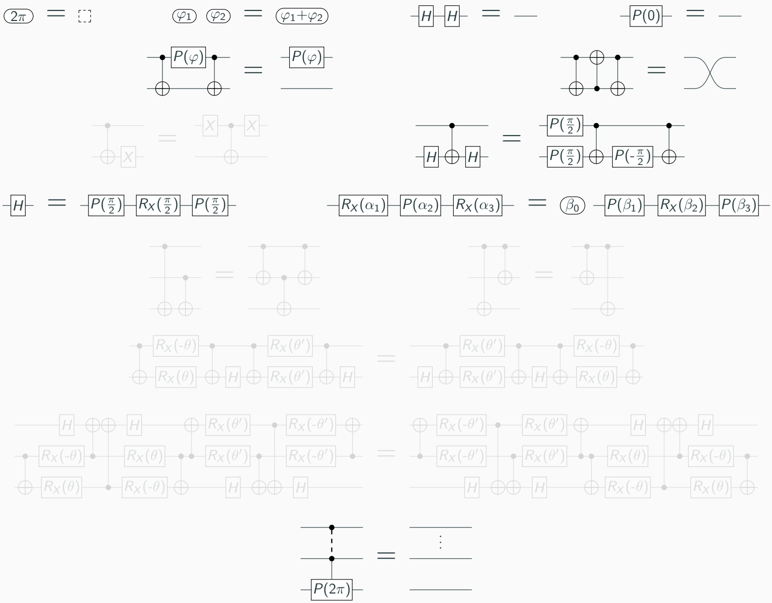
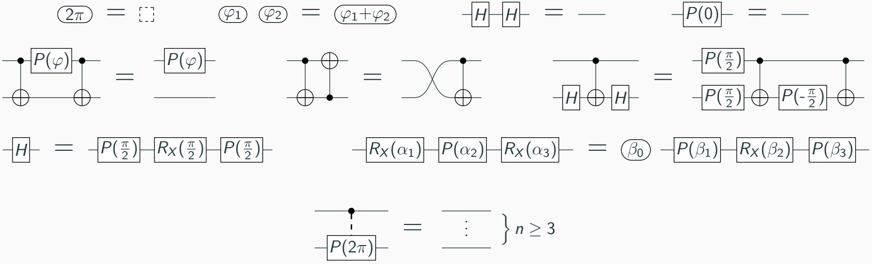


Towards the limit of simplifications



Question: Can we simplify the equational theory even more?

Theorem

This equational theory is complete, sound and minimal.

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All equations are independents.

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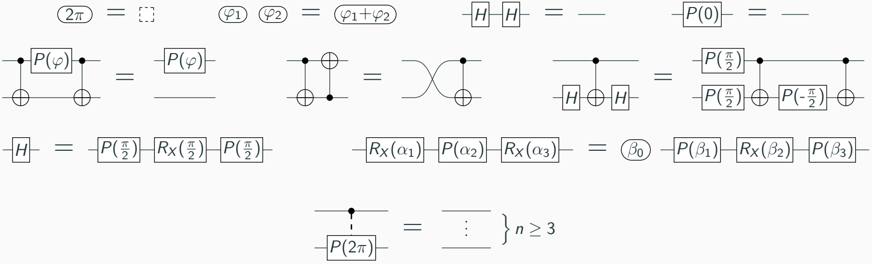

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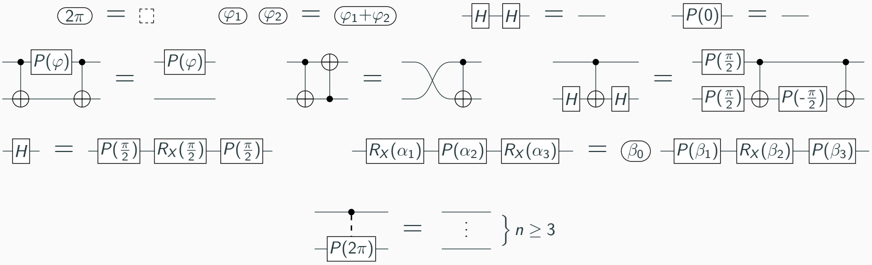

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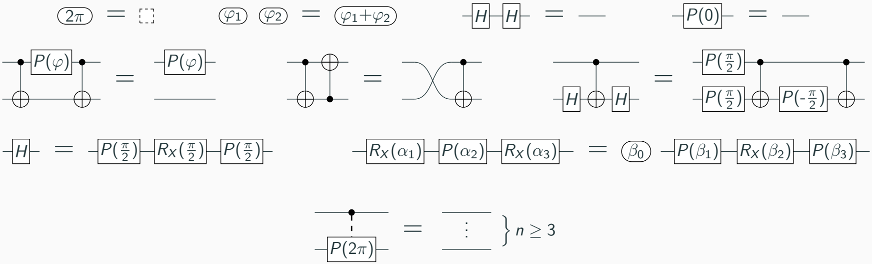

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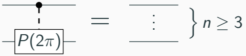
Unboundedness of the equational theory

Every instances of  $n \geq 3$ are necessary (for every $n \geq 3$).

Theorem

There is no complete equational theory for quantum circuits made of equations acting on a bounded number of wires.

More precisely, any complete equational theory for quantum circuits has at least one equation acting on n wires for any $n \in \mathbb{N}$.

Every instances of  are necessary (for every $n \geq 3$).

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Unboundedness of the equational theory

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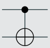
Alternative interpretation

For any $k \in \mathbb{N}$, for any quantum circuit C, let $[\![C]\!]_k^\sharp \in [0, 2\pi)$ be inductively defined as

$$[\![C_2 \circ C_1]\!]_k^\sharp = [\![C_1 \otimes C_2]\!]_k^\sharp = [\![C_2]\!]_k^\sharp + [\![C_1]\!]_k^\sharp \bmod 2\pi$$

$$[\![\,\fbox{}\,]\!]_k^\sharp = [\![\,-\!\!-\!\!-\,]\!]_k^\sharp = 0 \qquad [\![\,\textcircled{\varphi}\,]\!]_k^\sharp = 2^k\varphi \bmod 2\pi \qquad [\![\,-\boxed{H}-\,]\!]_k^\sharp = 2^{k-1}\pi \bmod 2\pi$$

$[\![$  $]\!]_k^\sharp = [\![$  $]\!]_k^\sharp = 2^{k-2}\pi \bmod 2\pi \qquad [\![\,-\boxed{P(\varphi)}-\,]\!]_k^\sharp = 2^{k-1}\varphi \bmod 2\pi$

Any sound equation involving circuits acting on at most $n-1$ wires is also sound according to $[\![\cdot]\!]_{n-1}^\sharp$.

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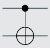
$[\![$  $]\!]^\sharp_k = [\![$  $]\!]^\sharp_k = 2^{k-2}\pi \bmod 2\pi \qquad [\![\,-\boxed{P(\varphi)}-\,]\!]^\sharp_k = 2^{k-1}\varphi \bmod 2\pi$

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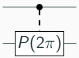
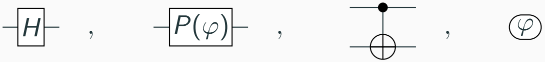
$[\![$  $\Big\}\, n \,]\!]^\sharp_{n-1} \;=\; \pi \;\neq\; 0 \;=\; [\![$  $\Big\}\, n \,]\!]^\sharp_{n-1}$

Extension to quantum circuits with ancillae

Quantum circuits with ancillae are generated by



together with

$$\vdash \quad \text{and} \quad \dashv$$

respectively denoting wire initialisation and wire termination.

(The generator $\dashv$ can only be applied to wires in the $|0\rangle$-state.)

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We extend $[\![\cdot]\!]$ with $[\![\vdash]\!] = |0\rangle$ and $[\![\dashv]\!] = \langle 0|$.

Universal for isometries

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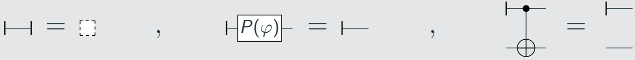
Theorem [CDPV CSL'24]

Adding those three equations makes the equational theory complete for quantum circuits with ancillae.



Using ancillae, we can build controlled gates without dividing the angles.

In these more general settings, is derivable for $n \geq 4$.

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Boundedness of the equational theory with ancillae

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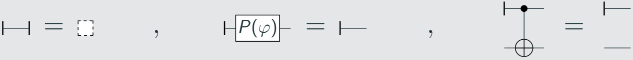

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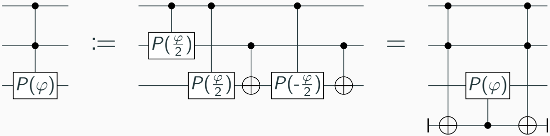

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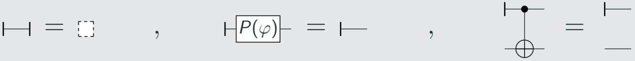

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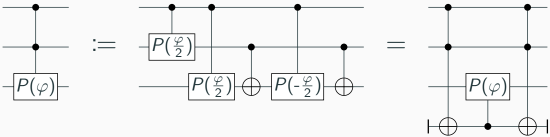
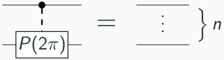


In these more general settings, 

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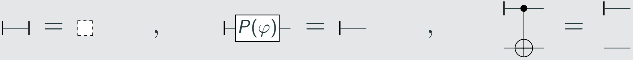

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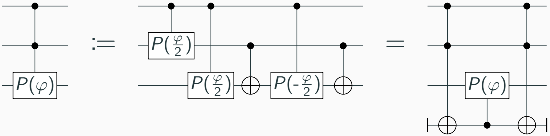
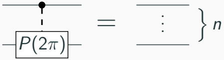


In these more general settings, 

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Thanks

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Minimal Equational Theories for Quantum Circuits

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