Minimal Equational Theories for Quantum Circuits

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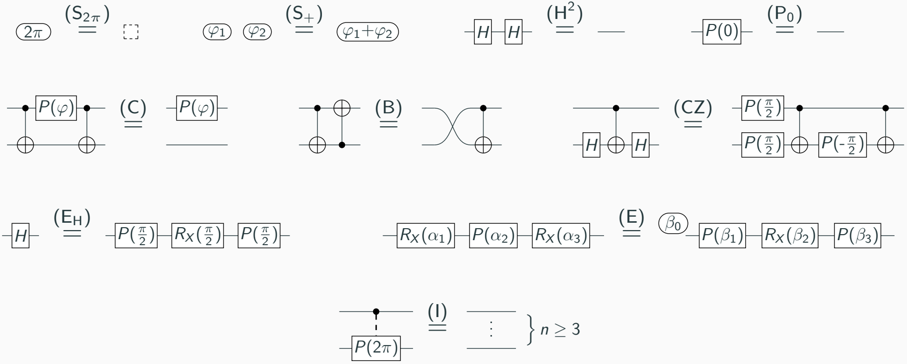
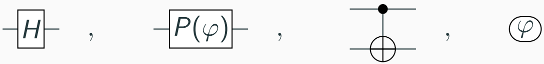
This is a minimal complete equational theory for quantum circuits.



Quantum circuits as a graphical language

Quantum circuits are generated by



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which come from the prop formalism[1] together with some deformation rules that ensure that circuits are defined "up to deformation".

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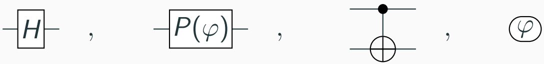

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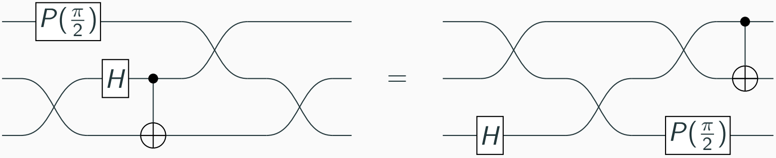

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Standard interpretation of quantum circuits

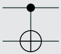
$$\text{circuits} \neq \text{unitaries}$$

Semantics

$$[\![C_2 \circ C_1]\!] = [\![C_2]\!] \circ [\![C_1]\!] \qquad [\![C_1 \otimes C_2]\!] = [\![C_1]\!] \otimes [\![C_2]\!]$$

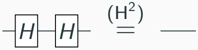
$[\![$  $]\!] = (1)$ $\qquad [\![\varphi]\!] = (e^{i\varphi})$

$[\![$ $]\!] = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ $\qquad [\![\boxed{H}]\!] = 1/\sqrt{2}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \qquad [\![\boxed{P(\varphi)}]\!] = \begin{pmatrix} 1 & 0 \\ 0 & e^{i\varphi} \end{pmatrix}$

$[\![$  $]\!] = \begin{pmatrix} 1&0&0&0 \\ 0&1&0&0 \\ 0&0&0&1 \\ 0&0&1&0 \end{pmatrix}$ $\qquad [\![$  $]\!] = \begin{pmatrix} 1&0&0&0 \\ 0&0&1&0 \\ 0&1&0&0 \\ 0&0&0&1 \end{pmatrix}$

Using axioms to transform circuits

We can use simple axioms such that,



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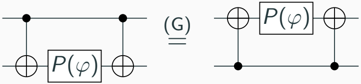

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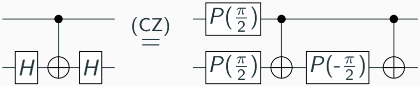

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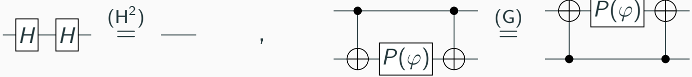

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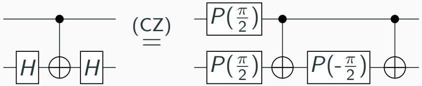

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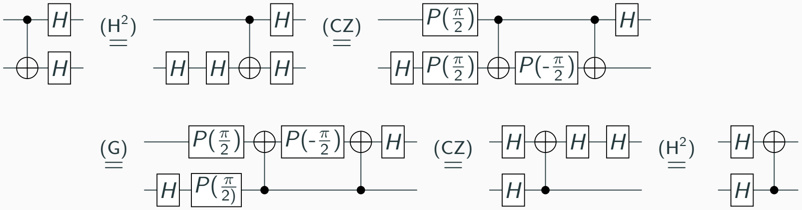

Desired properties for equational theories

Question: Is there an equational theory (i.e. a set of axioms) from which we can derive any true equation and only true equations?

Soundness

Any derivable equation is true.

$$\forall C_1, C_2 \in \mathcal{QC} \quad : \quad \Gamma \vdash C_1 = C_2 \implies [\![C_1]\!] = [\![C_2]\!]$$

Completeness

Any true equation is derivable.

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Trivial, just take all sound equations.

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Real goal: find a small complete and sound equational theory.

$\longrightarrow$ `arXiv:2206.10577(LICS2023)`[2]

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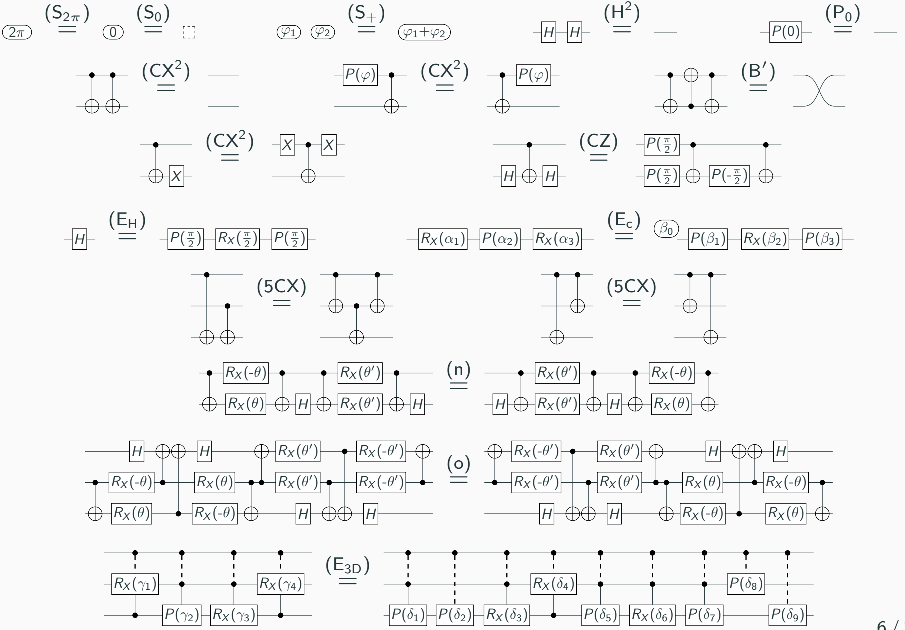
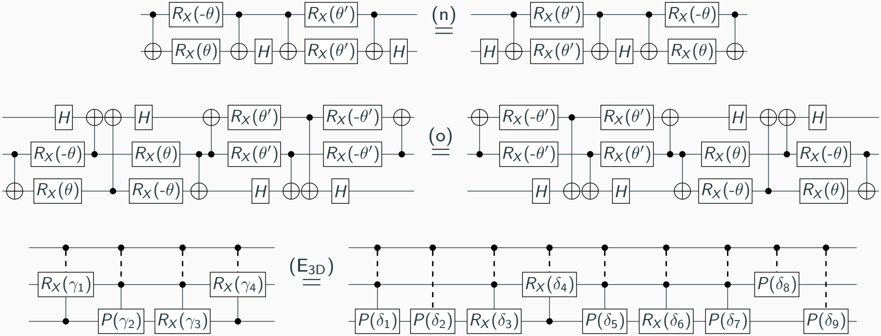
The first complete and sound equational theory



Problem: Many equations including non-intuitive and weird ones.



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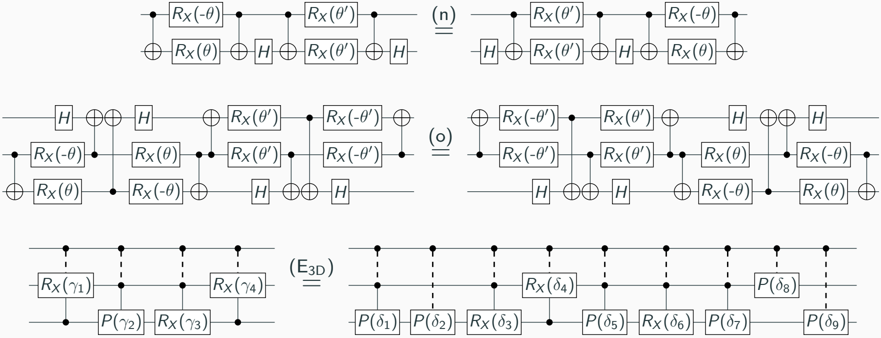

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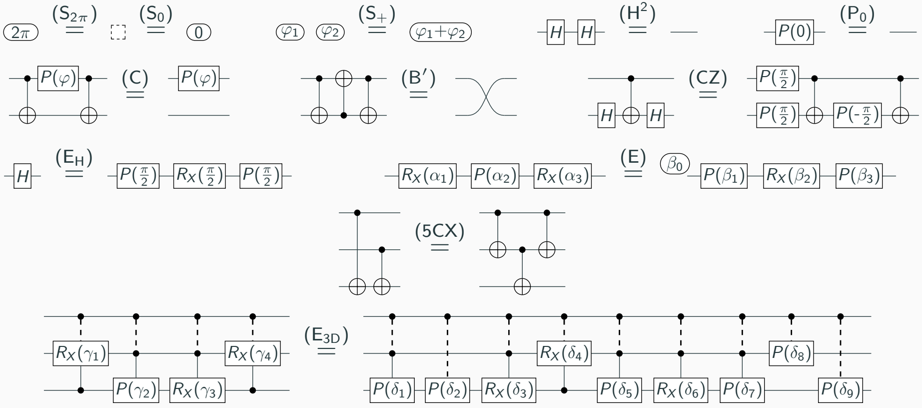
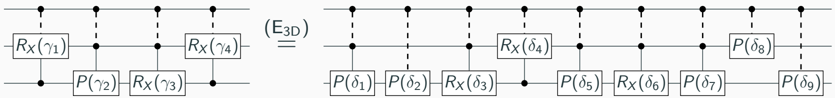
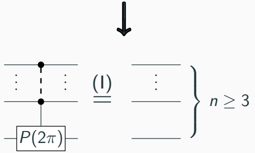
arXiv:2303.03117 (CSL2024)[3]



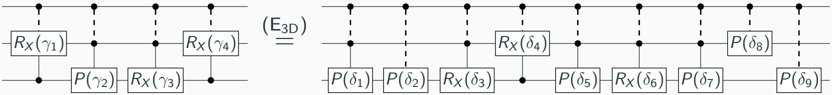
[3]Quantum Circuit Completeness: Extensions and Simplifications. Alexandre Clément, Noé Delorme, Simon Perdrix, Renaud Vilmart. CSL2024.

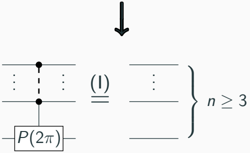
Killing the remaining weird rule





The two following intermediate results are the key to derive (E_{3D}) .





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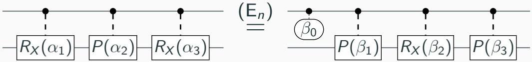

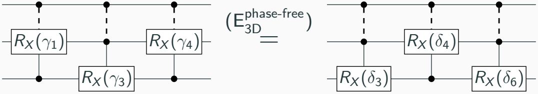

Towards the limit of simplifications

Question: Can we simplify the equational theory even more?

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All axioms are independents.

$$\forall (C_1 = C_2) \in \Gamma \quad : \quad \Gamma \backslash \{C_1 = C_2\} \nvdash C_1 = C_2$$

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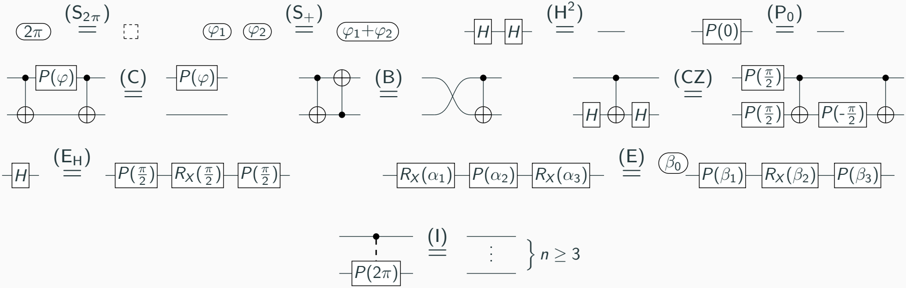
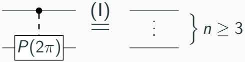
The minimal complete and sound equational theory

Theorem

This equational theory is complete, sound and minimal.



Every instances of  are necessary (for every $n \geq 3$).

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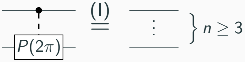
There is no complete equational theory for quantum circuits made of equations acting on a bounded number of qubits.

More precisely, any complete equational theory for n-qubit quantum circuits has at least one rule acting on n qubits.

Every instances of 

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Proof sketch of the main theorem

Alternative interpretation

For any $k \in \mathbb{N}$, for any quantum circuit C, let $[\![C]\!]^\sharp_k \in [0, 2\pi)$ be inductively defined as

$$[\![C_2 \circ C_1]\!]^\sharp_k = [\![C_1 \otimes C_2]\!]^\sharp_k = [\![C_2]\!]^\sharp_k + [\![C_1]\!]^\sharp_k \bmod 2\pi$$

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$$[\![\,\mathrm{CNOT}\,]\!]^\sharp_k = [\![\,\mathrm{SWAP}\,]\!]^\sharp_k = 2^{k-2}\pi \bmod 2\pi \qquad [\![\,-\boxed{P(\varphi)}-\,]\!]^\sharp_k = 2^{k-1}\varphi \bmod 2\pi$$

Any sound equation involving quantum circuits acting on at most $n-1$ qubits is also sound according to $[\![\cdot]\!]^\sharp_{n-1}$.

However,

$$\left[\!\!\left[\; \text{multi-controlled } -\boxed{P(2\pi)}- \;\right\}n \right]\!\!\right]^\sharp_{n-1} = \pi \neq 0 = \left[\!\!\left[\; \vdots \;\right\}n \right]\!\!\right]^\sharp_{n-1}$$

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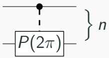
$\llbracket$  $\rrbracket_{n-1}^\sharp = \pi \neq 0 = \llbracket$  $\rrbracket_{n-1}^\sharp$

Discussion of the theorem

Possible weakness: $[\![C]\!]^\sharp_k$ is closely related to the determinant of $[\![C]\!]$. What if we consider quantum circuits up to global phases?

$\longrightarrow$ The theorem still holds!

Possible weakness: The choice of the generators $-\boxed{H}-$, $-\boxed{P(\varphi)}-$, , φ is not unique. What if we take another univeral gate set?

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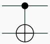
Possible weakness: The choice of the generators $\boxed{H}$, $\boxed{P(\varphi)}$,  , φ is not unique. What if we take another univeral gate set?

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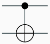
Possible weakness: The choice of the generators $\boxed{H}$, $\boxed{P(\varphi)}$,  , φ is not unique. What if we take another univeral gate set?

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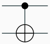
Possible weakness: The choice of the generators $-\boxed{H}-$, $-\boxed{P(\varphi)}-$,  , φ is not unique. What if we take another univeral gate set?

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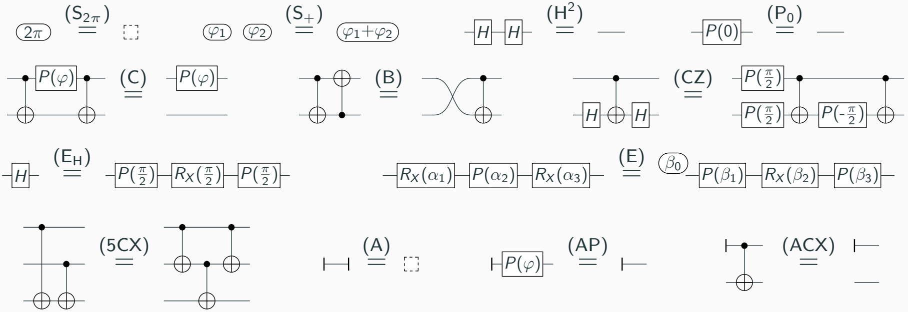
Quantum circuits with ancillae

Interestingly, there is a complete equational theory for quantum circuits with auxiliary qubits (universal for isometries) made of equations acting on a bounded number of qubits.



Thanks

arxiv:2311.07476